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Unilateral Crack Identification: A Filter-Driven, Iterative, Boundary Element Approach*

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Abstract. An identification problem for parametric variational inequalities and linear complementarity problems is solved here by means of iterative filter techniques. A concrete application in engineering mechanics, the unilateral crack identification problem, is solved. The elastic contact problem is formulated by boundary element-linear complementarity techniques. By means of numerical results and comparison with previous approaches based on optimization and neural networks it is shown that this method is advantageous. In view of the difficulty of the considered bilevel optimization problem, this approach may be of interest for other applications as well.

Key words: Complementarity problems; Extended Kalman filter; Identification; Unilateral contact; Variational inequalities

AMS: 73M25, 73T05

1. Introduction

Some inverse problems in engineering mechanics related to unilateral crack identification are solved in this paper by means of filter-driven iterative techniques, which include the well-known extended Kalman filter (EKF) approach combined with an appropriate Linear Complementarity-Boundary Element (LCP-BEM) method. It is shown by means of numerical examples that if this approach converges at all, which is almost always the case in the considered numerical examples, it is much more effective than more classical methods which are based on measurement error minimization with classical, genetic or backpropagation neural networks.

Contact relations introduce inequalities (no-penetration and compression-only requirements) and complementarity relations (either contact or separation) in mechanics. They constitute a typical application from the area of inequality or nonsmooth mechanics [30]. The mathematical problems which arise in this context are variational inequalities, complementarity and nonsmooth optimization problems. Mathematical programming and optimization techniques have already found their place in the computational mechanics' literature and they are used for the solution of

* This paper is dedicated to the memory of Professor P.D. Panagiotopoulos.

everyday practical applications, see, among others, [30, 3, 31, 32, 25, 8] and the references given there.

The classical approach for the study of structural identification, inverse problems in nonsmooth mechanics is to formulate them as output error minimization problems (cf., for classical problems [27]). The state relation is a parametric variational inequality or a parametric complementarity relation. Therefore, one gets problems of the so-called generalized bilevel optimization or optimization under equilibrium constraints [21, 34, 24, 29, 17]. The arising theoretical difficulties which are due to the nonclassical nature of the complementarity subsidiary condition, namely the nonconvexity and the nondifferentiability, are discussed in the above-cited publications. In addition, one has the ill-posedness of the inverse problem. In view of these difficulties, the study of effective numerical algorithms which are suitable for industrial applications is of importance.

The authors, based on previous positive experiences from the area of inverse problems in mechanics, consider the application of filter techniques in this paper. The concrete application concerns the identification of a unilateral crack, i.e., a crack which may close under the action of certain loading. Static loading is considered here. This problem has been recently studied in [36] by using a neural network technique and in [1] by means of optimization. Both papers use different boundary element techniques for the modelling of the mechanical problem and seems to be the first ones to deal with the inverse unilateral crack identification in elastostatics. The present paper is organized as follows. The nonsmooth mechanics' modelling of the unilateral crack problem is outlined in the next section. The filter algorithms and a short review of recent filter applications on inverse problems are discussed in section three. Numerical results are presented in the last section.

2. Modelling of the direct mechanical problem

Static and dynamic modelling of two-dimensional elastic structures with cracks or interfaces with and without unilateral contact effects is outlined in this section. More details on contact mechanics modelling can be found in [3] and on the unilateral crack application in [36, 37], respectively.

2.1. STATIC PROBLEMS

Let us consider a two-dimensional elastic structure. After appropriate discretization by the BEM, one obtains the following parametric matrix equation between boundary displacements $\mathbf{u}$ and boundary tractions $\mathbf{t}$ (cf. [7]):

$$\mathbf{H}(\mathbf{z})\mathbf{u} = \mathbf{G}(\mathbf{z})\mathbf{t}\,. \tag{1}$$

In (1), $\mathbf{u}$ is the vector of nodal displacements at the boundary, $\mathbf{t}$ is the boundary traction vector, and the nonsymmetric matrices $\mathbf{H}$ and $\mathbf{G}$ are appropriate influence matrices which are based on the used fundamental solution and the adopted boundary element discretization. All values which describe the shape of the plate, material constants, etc. form the parameter vector $\mathbf{z}$. In this work the so-called two-region BEM is used, as it is outlined in Figure 2, and the parameters $\mathbf{z}$ describe the position, the length and the inclination of a crack.

For classical boundary conditions, and for a given value of the parameter $\mathbf{z}$ (i.e., for a fixed crack with given position, length, etc.) known and unknown quantities of vectors $\mathbf{u}$ and $\mathbf{t}$ are separated, and all unknowns are gathered in the vector $\mathbf{x}$ and, finally, the system of equations:

$$\mathbf{A}(\mathbf{z})\mathbf{x} = \mathbf{b}(\mathbf{z}) \tag{2}$$

is formulated and solved.

For the parametric problem, one observes that the structural response $\mathbf{x}(\mathbf{z}) = \mathbf{A}(\mathbf{z})^{-1}\mathbf{b}(\mathbf{z})$ is, in general, a nonlinear relation of $\mathbf{z}$.

Let us further consider a structure with unilateral contact effects. This is the case of a crack which closes (partially or totally) under the action of an external loading. Using appropriate notation, let $\mathbf{u}_{cn}$, $\mathbf{u}_{ct}$, $\mathbf{t}_{cn}$, $\mathbf{t}_{ct}$ be the boundary nodal displacements and the boundary nodal tractions, respectively, along the normal (n) and the tangential (t) direction at the unilateral (contact) boundary. After partitioning, we arrive from (1) at:

$$\begin{bmatrix} \mathbf{H}_{ff}(\mathbf{z}) & \mathbf{H}_{fcn}(\mathbf{z}) & \mathbf{H}_{fct}(\mathbf{z}) \\ \mathbf{H}_{cf}(\mathbf{z}) & \mathbf{H}_{ccn}(\mathbf{z}) & \mathbf{H}_{cct}(\mathbf{z}) \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{u}_{cn} \\ \mathbf{u}_{ct} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_f \\ \mathbf{f}_c \end{bmatrix} + \begin{bmatrix} \mathbf{G}_{fcn}(\mathbf{z}) & \mathbf{G}_{fct}(\mathbf{z}) \\ \mathbf{G}_{ccn}(\mathbf{z}) & \mathbf{G}_{cct}(\mathbf{z}) \end{bmatrix} \begin{bmatrix} \mathbf{t}_{cn} \\ \mathbf{t}_{ct} \end{bmatrix} . \tag{3}$$

Here, the BEM equations are written for the two parts of each boundary. These groups correspond to the two rows of the supermatrices $\mathbf{H}$ and $\mathbf{G}$, i.e., the part of free (classical) boundaries (first subscript f) and the part of the contact interface (with first subscript c).

Finally, the vectors of boundary displacements and boundary tractions are composed of the parts which correspond to the bilaterally connected interface (with unknowns gathered in vector $\mathbf{x}$, as previously, and given values used for the construction of vector $\mathbf{f}$, cf. relation (2)) and the contributions normally and tangentially to the unilateral part (with subscripts cn and ct, respectively).

The requirements that along the unilateral contact boundary no penetration takes place, that only compression tractions are allowed and that either a contact is realized or separation with zero traction takes place, are expressed by means of the following inequalities and the complementarity relation:

$$\mathbf{u}_{cn} \leqslant \mathbf{0}, \mathbf{t}_{cn} \leqslant \mathbf{0}, \mathbf{u}_{cn}^T\mathbf{t}_{cn} = 0 . \tag{4}$$

Appropriate assumptions for the tangential components (cf. friction) completes the description of the mechanical problem. For instance, for a frictionless problem, one has $\mathbf{t}_{ct} = 0$ in (3).

Thus, for a fixed value of the parameter **z** one has the mechanical problem expressed in the form of a linear complementarity problem (LCP) composed of (3) and (4). For a variable **z** one has a parametric LCP which, after appropriate matricial manipulation and reduction of the free (unconstrained) unknowns, may be written as:

$$\mathbf{u}_{cn} = \mathbf{u}_{cn}^{(0)}(\mathbf{z}) + \mathbf{F}(\mathbf{z})\mathbf{t}_{cn} \leqslant \mathbf{0},\ \mathbf{t}_{cn} \leqslant \mathbf{0},\ \mathbf{u}_{cn}^T\mathbf{t}_{cn} = 0\ . \tag{5}$$

For completeness, one should mention that more general nonlinear complementarity problems (NCP) may also be considered instead of (5), i.e., problems of the form:

$$\mathbf{u}_{cn} = \mathscr{F}(\mathbf{z}, \mathbf{t}) \leqslant \mathbf{0},\ \mathbf{t}_{cn} \leqslant \mathbf{0},\ \boldsymbol{u}_{cn}^T\mathbf{t}_{cn} = 0\ . \tag{6}$$

For the inverse crack identification problems studied in this paper, NCP would arise in three-dimensional frictional contact problems or in nonlinear elasticity applications (e.g., for elastomeric parts undergoing large displacements and deformations). The techniques which are used in this paper are, in principle, applicable to this class of problems. Their implementation and testing remain open for future investigations.

The links between the theory of variational inequalities and of complementarity problems can be found in other publications and will not be discussed here (see, among others, [30, 28, 3, 23, 25]). It seems that in scientific applications which have a long tradition with variational methods (among others, in mechanics and in optimal control) the theory of variational inequalities is more familiar.

2.2. DYNAMIC PROBLEMS

In this case, after time discretization, for simplicity with constant time steps, the matrix formulation of the boundary element method in the kth time step reads (cf. [2, 13, 16]):

$$\mathbf{H}^{(1)}\mathbf{u}^{(k)} = \mathbf{G}^{(1)}\mathbf{t}^{(k)} + \sum_{m=1}^{k-1} [\mathbf{G}^{(k-m+1)}\mathbf{t}^{(m)} - \mathbf{H}^{(k-m+1)}\mathbf{u}^{(m)}]\ . \tag{7}$$

In (7), superscripts denote the time step in which the respective quantities are calculated. Moreover, matrix **H** is appropriately modified to take into account the dynamic effects, while the contribution of all previous steps is taken into account by the last terms in (7). Note that this formulation, as discussed for instance in [4, 13], is based on appropriate dynamic fundamental solutions.

2.3. PARAMETRIC SOLUTION MAPPING

It is well known that the solution mapping of the previous parametric LCP is, in general, a nonlinear, nondifferentiable and possibly multi-valued (set-valued) mapping which is denoted by:

$$\mathbf{u} = s(\mathbf{z}, \mathbf{t})\ . \tag{8}$$

For a fixed value of parameter **z**, one knows that mapping $s(\mathbf{z}, \mathbf{t})$ is piecewise-linear. Several properties of this mapping, including sensitivity results, are known for a fixed value of 'loading' vector **t** and when more information about the relations $\mathbf{H}(\mathbf{z})$, $\mathbf{G}(\mathbf{z})$ is available. For linear relations one may consult the book of Cottle et al. [12] and the finite element application in [35].

In this application, the loading vector **t** is kept constant and is given. Moreover, both the values of **u** and of its derivatives with respect to **z** are used. The derivatives should be considered in the generalized sense as set-valued multifunctions. Nevertheless, according to our previous numerical experience with optimal design for problems governed by variational inequalities [35], one may use in an approximate way only one element of this set-valued mapping without having convergence problems with several iterative algorithms. This is also the case with the error-insensitive Kalman filter, which is used in this paper. The required sensitivity information is obtained here numerically by simple forward-difference techniques.

3. Classical and extended Kalman filter and identification

3.1. REVIEW

Filters, in general, and in particular the EKF belong to the area of stochastic estimation and optimization. They are considered to be robust, error-insensitive signal processing methods. For details the reader may consult the original publications [18, 19] and, among others, [9] and the references in the other application-oriented papers cited later in the text. An EKF approach which copes with the parameter estimation problem for nonlinear systems is used in this paper for the solution of inverse, crack identification problems in mechanics.

It is worth noting that the effectiveness of stochastic approaches and of Kalman filters in engineering has been recognized early. See, in this respect, the calibration of structural elastoplasticity models [22, 6] and the geotechnical applications reported in [11].

Several applications of the Kalman filter method or of more advanced filtering techniques on inverse problems in engineering mechanics and structural analysis have been reported in recent publications. Most of them involve some automatic boundary or finite element routines for the successive re-analysis of the mechanical problem along the iteration steps of the filter. Thus, unknown circular or elliptical defects, partially combined with an unknown boundary shape for two-dimensional elastostatic applications modelled by the boundary element method are identified in [40]. Reference [42] deals with the identification of concentrated heat source in steady-state heat conduction field and with the identification of material properties and of unknown boundary values in two-dimensional isotropic elastostatics. EKF with boundary element methods are used in this work. A finite element-Kalman filter approach for the estimation of elastic material properties from measured surface displacements in elastostatics (geotechnical applications) is studied in [26]. Comparisons with earlier works and discussion on the effect of the off-diagonal terms of the error covariance matrix on the performance of the method are also given. An application on the identification of the material constants of the Gurson model for porous elastic-plastic materials using pseudoexperimental (finite element generated) data is reported in [5].

Some recent nonstructural applications of the EKF-BEM methodology are mentioned here, so that interested readers can find there more information. A calibration procedure for groundwater flow problems is studied in [15] by means of EKF techniques. Both steady state and transient groundwater flows are considered. They are modelled by means of the so-called dual reciprocity boundary element method (DRBEM). Groundwater flow identification problems are also treated in [14].

Boundary shape identification in steady state heat conduction problems [38] and noise source identification in acoustics [39] represent two more areas of successful recent applications. For other applications on inverse heat conduction problems the reader is referred to [33, 10] and [41]. The latter reference [41] contains an extension of the method for use with unknown input, which is also iteratively estimated within the proposed algorithm. Similar techniques may also be useful for more general health monitoring tasks in mechanics with unknown or partially known input (cf., the ambient vibration tests). In particular, the low computational cost of this method makes it promising for on-line monitoring and identification applications. These problems will not be discussed in this paper.

Finally, it should be mentioned that Kalman filter approaches may be used in connection with other advanced signal processing techniques to enhance their effectiveness or to enlarge the area of their applicability (see, among others, [20]).

The Kalman filter was originally derived for linear systems [18,19] but it was subsequently extended to cover nonlinear systems as well by means of a local linearization strategy, as it is outlined in the sequel.

3.2. DESCRIPTION

Let us consider that one has some measurements (i.e., observations) $\mathbf{y}$ of the parametrized mechanical problem (8). In mechanics, one usually measures displacements, strains, stresses or their time derivatives, so that one has, in general, the relation $\mathbf{y} = f(\mathbf{u})$, where $f(.)$ is an appropriate linear or nonlinear function. For notational simplicity, let us assume that some elements of vector $\mathbf{u}$ are measured, so that we can write $\mathbf{y} = \bar{s}(\mathbf{u})$ with obvious notation (cf. (8)).

For a given value $\mathbf{z}_k$ of parameter $\mathbf{z}$ the observation is written in the form:

$$\mathbf{y}_k = \bar{s}_k(\mathbf{z}_k, \mathbf{t}) + \nu_k \tag{9}$$

where ν_k is the observation noise, which is considered to be a white noise with zero mean value and of known covariance Q.

The plant of the system is, in general

$$\mathbf{z}_{k+1} = F_k \mathbf{z}_k + w_k \tag{10}$$

where w_k is the plant noise. In the estimation of static values studied in this paper, the plant dynamics simplify into:

$$\mathbf{z}_{k+1} = \mathbf{I}\mathbf{z}_k \,. \tag{11}$$

The filter-driven identification algorithm has the following steps:

Step 0. Set iteration counter $k = 0$ and choose initial value of the parameters z_0 and a given 'covariance' matrix Q.

Step 1. Iteration $k = k + 1$
the available observation (9) is assumed in the linearized form:

$$\mathbf{y}_k = s_k(\mathbf{z}_k, \mathbf{t}) + \nu_k = \mathbf{S}_k \mathbf{z}_k + \{s_k(\mathbf{z}_{k-1}, \mathbf{t}) - \mathbf{S}_k \mathbf{z}_{k-1}\} + \nu_k \tag{12}$$

where the linearization matrix at step k reads:

$$\mathbf{S}_k(\mathbf{z}_{k-1}) = \left(\frac{\partial s}{\partial z}\right)_{z = z_{k-1}} .$$

Only matrix $\mathbf{S}_k$ is calculated by using the estimate of $\mathbf{z}$ which is available from the previous step.

Step 2. The new estimation of the state, i.e., the new value of $\mathbf{z}$ is given by:

$$\mathbf{z}_k = \mathbf{z}_{k-1} + \mathbf{B}_k(\mathbf{y}_k - s(\mathbf{z}_{k-1}, \mathbf{t})) \tag{13}$$

where $\mathbf{y}_k$ is the wished values of the observation (measurements) and $s(\mathbf{z}_{k-1}, \mathbf{t})$ is the attained value of the observation (measurements) based on the estimation of the state values available from the previous iteration step. Moreover, matrix $\mathbf{B}_k$ is the filter gain, which may be calculated using Kalman filtering theory or other filtering techniques. The Kalman filter gain reads:

$$\mathbf{B}_k = \mathbf{R}\mathbf{S}_k^T(\mathbf{S}_k \mathbf{R} \mathbf{S}_k^T + \mathbf{Q})^{-1} \,, \tag{14}$$

where $\mathbf{R}$ is the error covariance of the measurements or an estimate of it. Nevertheless the following relatively simple projection filter [40] leads also to good results for the studied problem:

$$\mathbf{B}_k = (\mathbf{S}_k^T \mathbf{Q}^+ \mathbf{S}_k)^+ \mathbf{S}_k^T \mathbf{Q}^T \,, \tag{15}$$

where superscript $+$ denotes the Moore-Penrose generalized inverse of a matrix.

Step 3. If $\mathbf{z}$ converges then stop, otherwise continue iterations with Step 1.

The comment about the possibly multivaluedness, which was made in the introduction, should be considered in the interpretation of the sensitivity matrix $\mathbf{S}_k$.

For the numerical examples presented here, the simple filter gain (15) gave satisfactory results. Moreover, the choice of a value for the matrix $\mathbf{Q}$ (and $\mathbf{R}$, if applicable) does not influence the results. More complicated variants of the above basic algorithm with adaptively defined **Q** have been proposed in the literature and may be advantageous for other classes of problems.

4. Numerical experiments

4.1. PLATE WITH A CRACK

Some indicative results of static analyses of a plate which contains a crack are first presented. The geometrical data and the two-region BEM formulation of the problem are given in Figure 1. A crack of a length equal to 10.0 is placed in an orthogonal plate with dimensions 100.00×100.00. The center position of the crack is at the point with coordinates 50.0, 50.00. The plate is made of an elastic material with shear modulus equal to $G = 100000.0$ and Poisson's ratio equal to $\nu = 0.3$. All quantities are given in compatible units. Uniform loading is considered on the upper (free) boundary of the plate, while the lower boundary is fixed (supported).

The two-region boundary element discretization is shown in Figure 2. An indicative result for a horizontal loading along the upper part of the plate is shown in Figure 3 (deformation) and Figure 4 (vertical $t-x$ and horizontal $t-y$ components of the boundary and interface tractions). For this loading the crack is open.

4.2. CRACK IDENTIFICATION

The inverse problem consists in finding the position, the length and the orientation

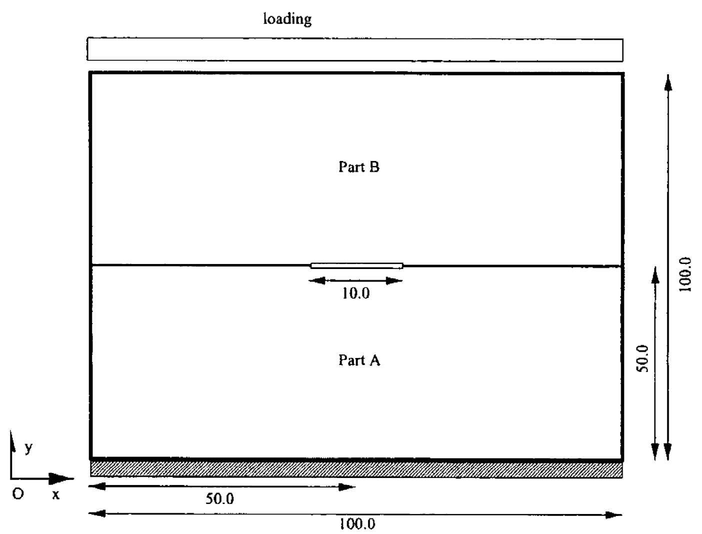

Figure 1. Configuration of the elastic plate with a crack.

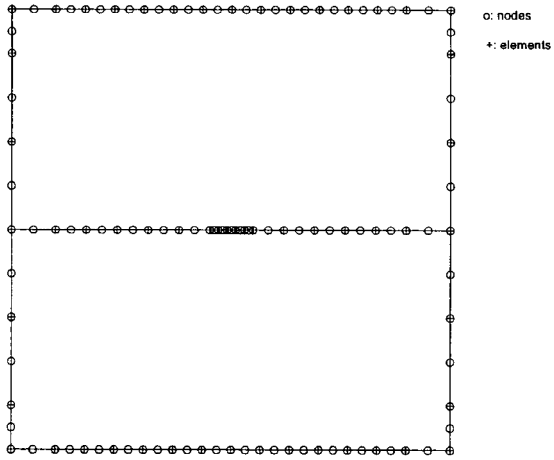

Figure 2. Boundary element discretization.

of a rectilinear crack contained in the plate of Figure 1. Both loading and measurement points are considered to lie on the upper part of the plate. A horizontal crack of length 10.0 with center at the position with horizontal, x-coordinate 44.0 and vertical, y-coordinate 68.00 is assumed. For all results a unit matrix $\mathbf{Q} = \mathbf{I}$ is used in the filter-driven algorithm with the projection filter gain (15). Convergence is assumed if the difference between two successive values of $\mathbf{z}$ is less than $1.e - 3$. In the sequel the crack will be moved in the plate so that its center will take positions between horizontal x-coordinate 30.0 and 70.0 and vertical y-coordinate 30.0 and 70.0. For each test position of the crack, the complete mechanical problem is solved automatically by the computer.

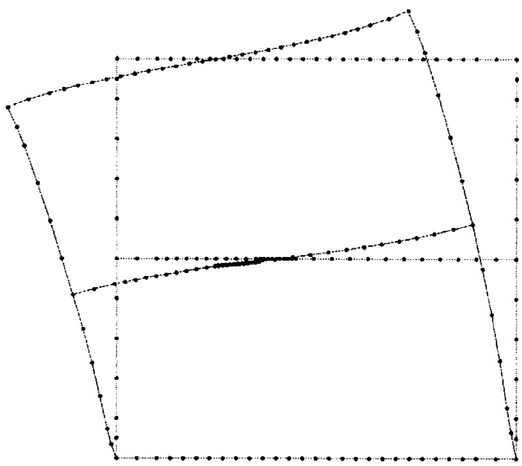

Figure 3. Initial and deformed configuration.

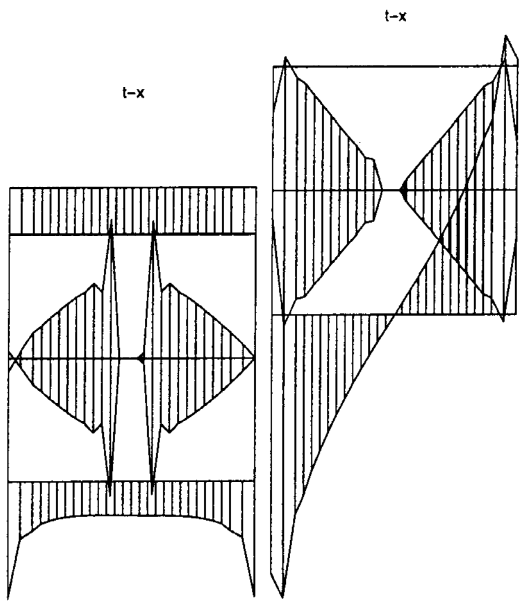

Figure 4. Boundary and interface tractions for the static crack analysis problem.

Let us consider that all displacements at the upper boundary of the plate, i.e., 60 values from the 30 boundary nodes of the upper boundary can be measured. Convergence of the algorithm depends on the used test loading, which also determines the state of the unilateral crack (i.e., open, partially closed, etc.). Four **loading cases** have been considered:

A vertical loading component equal to +100 (i.e., upwards or tensile loading for the plate)

B vertical loading component equal to −100 (i.e., downwards or compressive loading for the plate)

C horizontal loading component equal to +100 (i.e., from the left to the right) and vertical loading component equal to +100

D horizontal loading component equal to −100 and vertical loading component equal to +100

Loading cases **B** and **C** lead to partially or fully closed cracks for the most considered positions of the crack.

For the crack one may assume:

c classical crack (closes without contact) or

u unilateral crack (closes as in reality and transmits tractions) but without friction (small slips between the crack faces are allowed for).

The results of our numerical experiments are summarized as follows:

For every starting point with crack center coordinates between 30.0 and 70.0 the algorithm was able to find the correct position of the crack, provided that a consistent set of measurements and model is used. This means, for example, that one

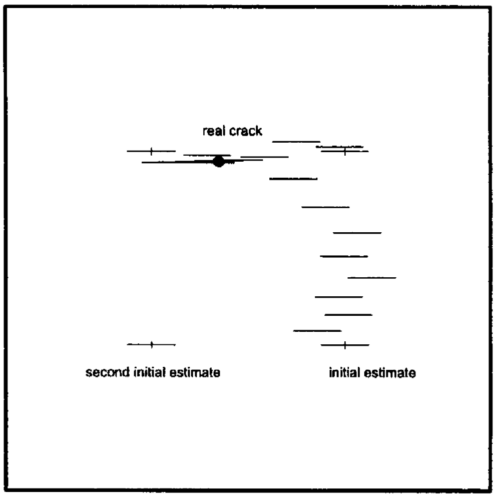

Figure 5. Convergence of the crack identification from four different starting points.

provides measurements for a partially closed crack, and uses the unilateral crack model for the inverse problem.

The computational cost is different and depends on the applied test loading. For loading cases **A** and **B**, the algorithm converged in 4–7 iterations. For loading cases **C** and **D**, the job was relatively complicated. A maximum step length (change in **z**) of 10 per cent of the value has been set to avoid nonconvergence. The correct solution was found after 3 to 14 iterations. As it is expected, less iterations are needed if a good initial estimate of the unknown values (starting point of the algorithm) is available. Five iterations with our 'home-made' FORTRAN programmes take about 7 minutes of computing time on an IBM RISC 6000 workstation. Sample results from the iterations of the algorithm are shown in Figures 5 and 6.

Addition of a reasonable random error in the measurements, up to 5 per cent, or consideration of less measurement points did not change the picture.

A more interesting result arises if one considers real-life measurements, i.e., measurements taken from the unilateral crack model **u** and tries to identify the crack by using a classical crack model **c**. In this case, the same previously tested algorithm systematically converged to inaccurate predictions. Thus, for loading case **B**, the algorithm predicts a crack with a center at point (53.30, 67.10), instead of the right one which lies at (44.0, 68.0). For loading case **D**, the algorithm converged to the point (44.50, 68.18), respectively. It should be noted that the previous values do not change essentially for all considered starting points.

5. Conclusions

For inverse problems, the use of the right, sufficiently accurate model is essential.

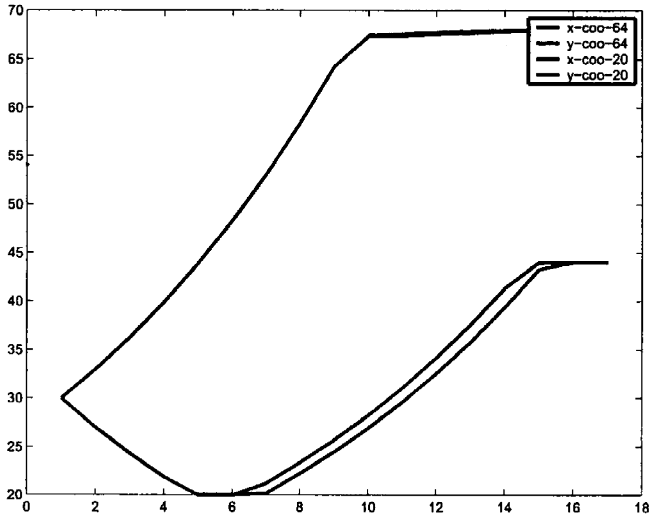

Figure 6. Convergence of the crack identification. Coordinates of the center of the crack.

Unilateral contact effects arising in cracks, or, in a more general context in interfaces between different parts or materials (e.g., in composites) should be taken into account.

Filtering algorithms seems to be much more effective than optimization based techniques. The same problem has been previously considered by the authors by means of neural network techniques [36] and by means of classical mathematical optimization algorithms (unpublished results). The convergence performance and the computational effort of the most simple filter algorithm used here are clearly better than the ones of the two previously mentioned techniques.

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